

Linear Matrix Inequalities for Ultimate Boundedness of Dynamical Systems with Conic Uncertain/Nonlinear Terms

Behçet Açıkmeşe

Department of Aerospace Engineering and Engineering Mechanics

The University of Texas at Austin, USA

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Abstract

This note introduces a sufficient Linear Matrix Inequality (LMI) condition for the ultimate boundedness of a class of continuous-time dynamical systems with conic uncertain/nonlinear terms.

1 Introduction

This note introduces an LMI [1] result for the ultimate boundedness of dynamical systems with conic uncertain/nonlinear terms. Earlier research developed necessary and sufficient conditions for quadratic stability for systems with similar characterizations for uncertainties and nonlinearities [2, 3, 4, 5, 6, 7]. Incremental version of these characterizations are used in the synthesis of nonlinear observers [8, 9, 10, 11, 12, 13, 14] and to design robust Model Predictive Control (MPC) algorithms [15, 16, 17, 18]. The following results first appeared in [17].

Notation: The following is a partial list of notation used in this paper: $Q=Q^T > (\geq) 0$ implies Q is a positive-(semi-)definite matrix; $Co\{G_1, \dots, G_N\}$ represents the convex hull of matrices G_1, \dots, G_N ; \mathbb{Z}^+ is the set of non-negative integers; $\|v\|$ is the 2-norm of the vector v ; $\lambda_{max}(P)$ and $\lambda_{min}(P)$ are maximum and minimum eigenvalues of symmetric

matrix P ; $\mathcal{E}_P := \{x : x^T P x \leq 1\}$ is an ellipsoid (possibly not bounded) defined by $P = P^T \geq 0$; for a bounded signal $w(\cdot)$, $\|w\|_{[t_1, t_2]} := \sup_{\tau \in [t_1, t_2]} \|w(\tau)\|$; for a compact set Ω , $\text{diam}(\Omega) := \max_{x, y \in \Omega} \|x - y\|$ and $\text{dist}(a, \Omega) := \min_{x \in \Omega} \|a - x\|$; and, for $V : \mathbb{R}^N \rightarrow \mathbb{R}$, $\nabla V = [\partial V / \partial x_1 \dots \partial V / \partial x_n]$. A set Ω is said to be *invariant* over $[t_0, \infty)$ for $\dot{x} = f(x, t)$ if: $x(t_0) \in \Omega$ implies that $x(t) \in \Omega$, $\forall t \geq t_0$. Ω is also *attractive* if for every $x(t_0)$, $\lim_{t \rightarrow \infty} \text{dist}(x(t), \Omega) = 0$.

2 A General Analysis Result on Ultimate Boundedness

The following lemma gives a Lyapunov characterization for the ultimate boundedness of a nonlinear time-varying system, which is used in the proof of main result.

Lemma 1. *Consider a system with state η and input σ described by*

$$\dot{\eta} = \phi(t, \eta, \sigma), \quad t \geq t_0. \quad (1)$$

Suppose there exists a positive definite symmetric matrix P with, $V(\eta) = \eta^T P \eta$, and a continuous function W such that for all η , σ and $t \geq 0$

$$\dot{V} = 2\eta^T P \phi(t, \eta, \sigma) \leq -W(\eta) < 0 \quad \text{when} \quad \eta^T P \eta > \|\sigma\|^2. \quad (2)$$

Then for every bounded continuous input signal $\sigma(\cdot)$, the ellipsoid $\mathcal{E} := \{\eta : \eta^T P \eta \leq \|\sigma(\cdot)\|_{[t_0, \infty)}^2\}$ is invariant and attractive for system (1). Furthermore, for any solution $\eta(\cdot)$ we have

$$\limsup_{t \rightarrow \infty} [\eta(t)^T P \eta(t)] \leq \|\sigma(\cdot)\|_{[t_0, \infty)}^2. \quad (3)$$

See [19] for a proof of the above lemma.

3 Analysis of Systems with Conic Uncertainty/ Nonlinearity

In this section we consider the following system

$$\dot{x} = Ax + Ep(t, x) + Gw \quad (4)$$

where x is the state, p represents the uncertain/nonlinear terms, and w is a bounded disturbance signal, and $p \in \mathcal{F}(\mathcal{M})$ with

$$q = C_q x + Dp. \quad (5)$$

To define $p \in \mathcal{F}(\mathcal{M})$, let

$$\mathcal{F}(\mathcal{M}) := \left\{ \phi : \mathbb{R}^{n_q+1} \rightarrow \mathbb{R}^{n_p} : \phi \text{ satisfies QI (7)} \right\}. \quad (6)$$

where the following QI (*Quadratic Inequality*) is satisfied

$$\begin{bmatrix} q \\ \phi(t, v) \end{bmatrix}^T M \begin{bmatrix} q \\ \phi(t, v) \end{bmatrix} \geq 0, \quad \forall M \in \mathcal{M}, \quad \forall v \in \mathbb{R}^{n_q}, \text{ and } \forall t. \quad (7)$$

where \mathcal{M} is a set of symmetric matrices.

The following condition, which is instrumental in the control synthesis, is assumed to hold for the incrementally-conic uncertain/nonlinear terms.

Condition 1. *There exist a nonsingular matrix T and a convex set \mathcal{N} of matrix pairs (X, Y) with $Y \in \mathbb{R}^{n_p \times n_p}$ and X, Y symmetric and nonsingular such that for each $(X, Y) \in \mathcal{N}$, the matrix*

$$M = T^T \begin{bmatrix} X^{-1} & 0 \\ 0 & -Y^{-1} \end{bmatrix} T \in \mathcal{M} \quad \text{with} \quad T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}, \quad (8)$$

where $T_{22} + T_{21}D$ is nonsingular, $T_{21} \in \mathbb{R}^{n_p \times n_q}$ and $T_{22} \in \mathbb{R}^{n_p \times n_p}$. Furthermore, the set \mathcal{N} can be parameterized by a finite number of LMIs.

It is also assumed that the set of multipliers \mathcal{M} satisfies Condition 1. The following theorem, the main result of this note, presents an LMI condition guaranteeing ultimate boundedness of all the trajectories of the system (4).

Theorem 1. *Consider the system given by (4) with $p \in \mathcal{F}(\mathcal{M})$ where the multiplier set \mathcal{M} satisfies Condition 1. Suppose that there exist $Q = Q^T > 0$, $(X, Y) \in \mathcal{N}$, $\lambda > 0$, and $R = R^T > 0$ such that the following matrix inequality holds*

$$\begin{bmatrix} (A - E\Gamma^{-1}T_{21}C_q)Q + Q(A - E\Gamma^{-1}T_{21}C_q)^T + \lambda Q + R & E\Gamma^{-1}Y & QC_q^T\Sigma^T & G \\ Y\Gamma^{-T}E^T & -Y & Y\Lambda^T & 0 \\ \Sigma C_q Q & \Lambda Y & -X & 0 \\ G^T & 0 & 0 & -\lambda I \end{bmatrix} \leq 0 \quad (9)$$

where

$$\Gamma = T_{21}D + T_{22}, \quad \Lambda = (T_{11}D + T_{12})\Gamma^{-1}, \quad \Sigma = T_{11} - (T_{11}D + T_{12})\Gamma^{-1}T_{21}.$$

Then, letting $V(x) := x^T Q^{-1}x$, we have

$$\dot{V}(x) + x^T Q^{-1} R Q^{-1} x \leq 0, \quad \forall V(x) \geq \|w\|^2. \quad (10)$$

Proof. First pre- and post-multiply (9) by

$$\begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \\ 0 & 0 & I & 0 \end{bmatrix}$$

and then pre- and post-multiply the resulting matrix inequality with $\text{diag}(Q^{-1}, Y^{-1}, I, I)$ to obtain

$$\begin{bmatrix} \begin{pmatrix} Q^{-1}(A - E\Gamma^{-1}T_{21}C_q) + (A - E\Gamma^{-1}T_{21}C_q)^T Q^{-1} \\ + \lambda Q^{-1} + Q^{-1} R Q^{-1} \\ \Gamma^{-T} E^T Q^{-1} \\ G^T Q^{-1} \\ \Sigma C_q \end{pmatrix} & Q^{-1} E \Gamma^{-1} & Q^{-1} G & C_q^T \Sigma^T \\ -Y^{-1} & 0 & \Lambda^T & \\ 0 & -\lambda I & 0 & \\ \Lambda & 0 & -X & \end{bmatrix} \leq 0$$

By using Schur complements the above inequality implies that

$$\begin{bmatrix} Q^{-1}(A - E\Gamma^{-1}T_{21}C_q) + (A - E\Gamma^{-1}T_{21}C_q)^T Q^{-1} + \lambda Q^{-1} + Q^{-1} R Q^{-1} & Q^{-1} E \Gamma^{-1} & Q^{-1} G \\ \Gamma^{-T} E^T Q^{-1} & -Y^{-1} & 0 \\ G^T Q^{-1} & 0 & -\lambda I \end{bmatrix} + \begin{bmatrix} \Sigma C_q & \Lambda & 0 \end{bmatrix}^T X^{-1} \begin{bmatrix} \Sigma C_q & \Lambda & 0 \end{bmatrix} \leq 0,$$

which then implies that

$$\begin{bmatrix} Q^{-1}(A - E\Gamma^{-1}T_{21}C_q) + (A - E\Gamma^{-1}T_{21}C_q)^T Q^{-1} + \lambda Q^{-1} + Q^{-1} R Q^{-1} & Q^{-1} E \Gamma^{-1} & Q^{-1} G \\ \Gamma^{-T} E^T Q^{-1} & 0 & 0 \\ G^T Q^{-1} & 0 & -\lambda I \end{bmatrix} + \begin{bmatrix} \Sigma C_q & \Lambda & 0 \\ 0 & I & 0 \end{bmatrix}^T \begin{bmatrix} X^{-1} & 0 \\ 0 & -Y^{-1} \end{bmatrix} \begin{bmatrix} \Sigma C_q & \Lambda & 0 \\ 0 & I & 0 \end{bmatrix} \leq 0.$$

Note that

$$\begin{bmatrix} \Sigma C_q & \Lambda & 0 \\ 0 & I & 0 \end{bmatrix} = \begin{bmatrix} \Sigma & \Lambda \\ 0 & I \end{bmatrix} \begin{bmatrix} C_q & 0 & 0 \\ 0 & I & 0 \end{bmatrix}.$$

Now post- and pre-multiply the earlier matrix inequality with the following matrix and its transpose

$$\begin{bmatrix} I & 0 & 0 \\ T_{21}C_q & \Gamma & 0 \\ 0 & 0 & I \end{bmatrix}$$

to obtain

$$\begin{bmatrix} Q^{-1}A + A^T Q^{-1} + \lambda Q^{-1} + Q^{-1}RQ^{-1} & Q^{-1}E & Q^{-1}G \\ E^T Q^{-1} & 0 & 0 \\ G^T Q^{-1} & 0 & -\lambda I \end{bmatrix} + \begin{bmatrix} C_q & 0 & 0 \\ T_{21}C_q & \Gamma & 0 \end{bmatrix}^T \begin{bmatrix} \Sigma & \Lambda \\ 0 & I \end{bmatrix}^T \begin{bmatrix} X^{-1} & 0 \\ 0 & -Y^{-1} \end{bmatrix} \begin{bmatrix} \Sigma & \Lambda \\ 0 & I \end{bmatrix} \begin{bmatrix} C_q & 0 & 0 \\ T_{21}C_q & \Gamma & 0 \end{bmatrix} \leq 0,$$

where

$$\begin{bmatrix} \Sigma & \Lambda \\ 0 & I \end{bmatrix} \begin{bmatrix} C_q & 0 & 0 \\ T_{21}C_q & \Gamma & 0 \end{bmatrix} = \begin{bmatrix} T_{11}C_q & T_{11}D + T_{12} & 0 \\ T_{21}C_q & T_{21}D + T_{22} & 0 \end{bmatrix} = T \begin{bmatrix} C_q & D & 0 \\ 0 & I & 0 \end{bmatrix}.$$

By using Condition 1,

$$M = T^T \begin{bmatrix} X^{-1} & 0 \\ 0 & -Y^{-1} \end{bmatrix} T \in \mathcal{M}$$

This implies that, for some $M \in \mathcal{M}$, we have

$$\begin{bmatrix} Q^{-1}A + A^T Q^{-1} + \lambda Q^{-1} + Q^{-1}RQ^{-1} & Q^{-1}E & Q^{-1}G \\ E^T Q^{-1} & 0 & 0 \\ G^T Q^{-1} & 0 & -\lambda I \end{bmatrix} + \begin{bmatrix} C_q & D & 0 \\ 0 & I & 0 \end{bmatrix}^T M \begin{bmatrix} C_q & D & 0 \\ 0 & I & 0 \end{bmatrix} \leq 0.$$

Pre- and post-multiplying the above inequality with $[x^T \ p^T \ w^T]$ and its transpose and using $V = x^T Q^{-1}x$, we obtain

$$2x^T Q^{-1}(Ax + Ep + Gw) + x^T Q^{-1}RQ^{-1}x + \lambda(V - \|w\|^2) + \begin{bmatrix} q \\ p \end{bmatrix}^T M \begin{bmatrix} q \\ p \end{bmatrix} \leq 0, \quad \text{for all } \begin{bmatrix} x \\ p \\ w \end{bmatrix}.$$

Since $p \in \mathcal{F}(\mathcal{M})$ with $q = Cx + Dp$, by using the S-procedure [1], the above inequality implies that the system (4) satisfies: $\dot{V} \leq -x^T Q^{-1}RQ^{-1}x < 0, \quad \forall V \geq \|w\|^2.$ ■

The following corollary gives a matrix inequality condition for the quadratic stability of the system (4) (when $w = 0$), that is, existence of a quadratic Lyapunov function $V = x^T P x$ proving the exponential stability by establishing

$$\dot{V} + x^T Q^{-1} R Q^{-1} x \leq 0 \quad (11)$$

for all trajectories of the system (4). The proof of the lemma follows from a straight adaption of the proof of Theorem 1.

Corollary 1. *Consider the system given by (4) with $w \equiv 0$ and $p \in \mathcal{F}(\mathcal{M})$ where the multiplier set \mathcal{M} satisfies Condition 1. Suppose that there exist $Q = Q^T > 0$, $(X, Y) \in \mathcal{N}$ and $\lambda > 0$ such that the following matrix inequality holds*

$$\begin{bmatrix} (A - E\Gamma^{-1}T_{21}C_q)Q + Q(A - E\Gamma^{-1}T_{21}C_q)^T + R & E\Gamma^{-1}Y & Q C_q^T \Sigma^T \\ Y\Gamma^{-T}E^T & -Y & Y\Lambda^T \\ \Sigma C_q Q & \Lambda Y & -X \end{bmatrix} \leq 0 \quad (12)$$

where Γ , Σ , Λ are as given in Theorem 1. Then the system (4) is quadratically stable with a Lyapunov function $V = x^T Q^{-1} x$ and all the trajectories satisfy

$$V(x(t)) \leq V(x(t_0)), \quad \forall t \geq t_0, \quad (13)$$

$$\dot{V}(x) + x^T Q^{-1} R Q^{-1} x \leq 0, \quad \forall x. \quad (14)$$

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